STANDARD BASES OF $K-$SUBSPACES

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Algebraic Geometry And Its Application

Abdus Salam School of Mathematical Sciences

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Outline

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Standard Bases for Special Subspaces

Let $U, M, N \subseteq K[[\mathrm{x}]]^q$ such that $U = M + N$ where M is a $K[[\mathrm{x}]]-$module and N is $K-$module. How to compute a special set of generators for U?

v-standard basis

Let $U \subseteq K[[\mathrm{x}]]^q$ be a subspace of the K-vector space $K[[\mathrm{x}]]^q$ and $>$ a local monomial ordering. A subset $W \subseteq U$ is called standard basis of U if $\mathrm{L}(U) = \mathrm{L}(W)$. Here $\mathrm{L}(U)$ is the K-vector space generated by the leading monomials of U with respect to the ordering $>$.

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Let $N = \langle w_1, \ldots, w_s \rangle_K$ and $\{m_1, \ldots, m_l\}$ be the set of monomials of $K[[\mathrm{x}]]^q$ not being in $\mathrm{L}(M)$.

We compute

$$\mathsf{REDNF}(w_i|M) = \sum_{j=1}^{l} c_{ij} m_j,\ c_{ij} \in K$$

$(\bar{\mathrm{c}}_{ij})$ be the matrix obtained from (c_{ij}) in reduced row echelon form, i.e. the $\bar{c}_{ij}$ have the following properties: $\exists \quad j_1, \ldots, j_a$ such that $\bar{c}_{ij_i} = 1$, $\bar{c}_{ij} = 0$ if $j < j_i$, $i = 1, \ldots, a$ and $\bar{c}_{ij} = 0$ if $i > a$.

Proposition

$\mathrm{L}(U)$ is the K-vector space generated by the monomials of $\mathrm{L}(M)$ and $\{m_{j_1}, \ldots, m_{j_a}\}$.

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Corollary

Let G be a standard basis of M as $K[[\mathrm{x}]]-$module and $H := \{\mathrm{x}^\alpha g | g \in G, \alpha \in \mathbb{Z}^n_{\geq 0}\}$ and $L := \{\sum_{j=1}^{l} \bar{c}_{ij} m_j, i = 1, \ldots, a\}$. Then $H \cup L$ is a standard basis of U as K-vector space.

Definition

The pair (G, L) a standard basis of U, if G is a standard basis of M (as $K[[\mathrm{x}]]$–module) and $H \cup L$ is a standard basis for U (as K–vector space).

Corollary

Let G be a standard basis of M as $K[[\mathrm{x}]]-$module and $H := \{\mathrm{x}^\alpha g | g \in G, \alpha \in \mathbb{Z}^n_{\geq 0}\}$ and $L := \{\sum_{j=1}^{l} \bar{c}_{ij} m_j, i = 1, \ldots, a\}$. Then $H \cup L$ is a standard basis of U as K-vector space.

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Algorithm

Algorithm 1 vSTD

Input: $M \subseteq K[[\mathrm{x}]]^q$ be $K[[\mathrm{x}]]$–module, $N \subseteq K[[\mathrm{x}]]^q$ finite dimensional K–vector space , bound an integer

Output: (G, L) a standard basis of $U + \langle \mathrm{x} \rangle^{\text{bound}} K[[\mathrm{x}]]^q$

1: compute G a standard basis of $M + \langle \mathrm{x} \rangle^{\text{bound}} K[[\mathrm{x}]]^q$;
2: use Gaussian algorithm to compute a reduced row echelon form $L := \{L_1, \ldots, L_t\}$ of REDNF$(N|G) := \langle N_1, \ldots, N_s \rangle$ with respect to the ordering.
3: **return** (G, L);

[0] D. Afzal, S.Kanwal, G. Pfister: *classifyMapGerms*.lib. A SINGULAR 4-0-3 library for computing the standard basis of the tangent space at the orbit of an algebraice group action (2016). https://github.com/Singular/Sources/blob/spielwiese/ Singular/ LIB/classifyMapGerms.lib.

Outline

Motivation of Modular Computations

Let $f = x^5 + y^{11} + xy^9 + x^3y^9$ and $I = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$.

- The Gröbner basis with respect to the lexicographical ordering
  - $264627y^{39} + 26244y^{35} - 1323135y^{30} - 131220y^{26} + 1715175y^{21} + 164025y^{17} + 1830125y^{16}$
  - $12103947791971846719838321886393392913750065060875xy^8 - \ldots + 14793713967965590435357948972258591339027857296625y^{10}$
  - $40754032969602177507873137664624218564815033875x^4 + \ldots + 8150806593920435501574627532924843712963006775y^9$

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Idea of Modular Computation

- Given an ideal $I = \{f_1, \ldots, f_m\} \subseteq \mathbb{Q}[x]$.
- choose a set of primes P
- compute a standard basis G_p of $\langle f_{1,p} \ldots, f_{m,p} \rangle \subset \mathbb{Z}_p[x]$ for each $p \in P$.
- lift these modular basis $\{G_p\}_{p \in P}$ to a set of polynomials $G \subseteq \mathbb{Q}[x]$
- test if G is a standard basis of I.
- If the test fails enlarge P and continue.

Verification of Result

If the result is correct?

Verification needed:

- G is a standard basis.
- $I = \langle G \rangle$.

$I \supseteq \langle G \rangle$ is expensive.

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Verification Theorem

Modular $v-$Standard Bases[1]

Let M, N be as above, let p be a prime and G, $L \subseteq \mathbb{Q}[\mathrm{x}]^q$ such that $\mathrm{LM}(G) = \mathrm{LM}(G_p)$ and $\mathrm{LM}(L) = \mathrm{LM}(L_p)$. Assume that (G_p, L_p) is a reduced standard basis of $M_p + N_p$ and G is a standard basis of $\langle G \rangle_{\mathbb{Q}[[\mathrm{x}]]}$. Assume that $M \subseteq \langle G \rangle_{\mathbb{Q}[[\mathrm{x}]]}$ and $U \subseteq \langle G \rangle_{\mathbb{Q}[[\mathrm{x}]]} + \langle L \rangle_{\mathbb{Q}}$ then (G, L) is a reduced standard basis of U.

[1] D. Afzal , S. Kanwal and G. Pfister: Tangent Space at the Orbit of an Algebraic Group Action, to appear in Bull. Math. Soc. Sci. Math. Roumanie (2017)

Examples for $\nu-$Standard Bases

Example1

$$\text{bound} = 15, f_1 = x, f_2 = xy + y^5 + y^7, g_1 = f_1 + f_1f_2,$$
$$g_2 = f_2 + f_1f_2, f = \langle g_1, g_2 \rangle$$
$$\phi(x,y) = (x + y + x^3 + xy^3 + y^{11} + y^{14} + xy^{17},$$
$$y + y^2 + x^3 + 2x^3y + x^6 + y^{14} + 2y^{15} + 2x^3y^{14} + xy^{17} + 2xy^{18} +$$
$$2x^4y^{17} + y^{28} + 2xy^{31} + x^2y^{34})$$
$$F = \phi(f), M = \langle x, y \rangle \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle, N = F\mathbb{C}[[F]]^2$$

$v-$Standard Basis

Example	vStd	modVStd	modVStd0
1	4123	2942	52
2	–	532	324
3	–	3	8
4	2258	36	24
5	56	19	12
6	864	4	14

Thank you!